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AN ALGEBRAIC APPROACH TO A CALCULUS
OF FUNCTIONAL DIFFERENCES:
FIXED DIFFERENCES AND INTEGRALS

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An Algebraic Approach to A Calculus of Functional Differences: Fixed Differences and Integrals

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Abstract

We introduce a notion of functional differences in which the difference of a function f with respect to a function h is that function g that describes how the value of f changes when its argument is altered by h: $f[h(x)] = g[f(x)]$. We also introduce the inverse operation of functional integration and derive useful properties of both operations. The result is a calculus that facilitates derivation and reasoning about recursive programs. This is illustrated in a number of simple examples. The present report uses algebraic methods to establish preliminary results pertaining to *fixed differences,* that is, functional differences that do not depend on the value of the argument x.

1. Motivation

Simple recursive definitions often take the following form:

$$\begin{aligned} fx_0 &= y_0 \\ f(hx) &= g(fx), \text{ for } x \neq x_0 \end{aligned} \tag{1}$$

The assumption here is that an arbitrary domain value x can be reached by finitely many applications of h. That is, for all acceptable x there is an n such that $x = h^n x_0$. More general patterns of recursive definition will be considered in Section 6 and more generally in [MacLennan '88].

Many of the results in this report are included in [MacLennan '86], however, that report used methods from the calculus of relations [Carnap '58]. The present report obtains the same results more easily by use of the methods of abstract algebra. The reason that the algebraic approach is easier seems to be that the algebraic notion of a function, which incorporates the domain and

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codomain as part of its definition, automatically takes care of issues that must otherwise be handled explicitly. In particular, the algebraic notation (and its assumption that all functions are total) simplifies reasoning about the domains of functions. Comparing the derivations in [MacLennan '86] and the present report is illuminating in this regard. The algebraic approach is essential for the investigation of variable differences and their associated integrals that will be presented in a later report [MacLennan '88]; for now we restrict our attention to fixed differences.[1]

In deriving a recursive definition for a particular f, there are four unknowns that must be found, g, h, x_0 and y_0. Since h and x_0 are usually determined by the domain in question (e.g., they are zero and the successor function for the domain of natural numbers), and y_0 is usually easily determined from the definition of f, the main problem is determining the function g.

To see how this can be done consider the second line in Eq. 1:

$$f\ (h\ x)\ =\ g\ (f\ x)$$

The function g tells us how much the value of the function f changes when its argument is changed by h. That is, if f's argument is changed by h, then its value is changed by g. This equation is analogous to the finite difference equation

$$f\ (h + x)\ =\ g + (f\ x)$$

The difference is that in the first equation the "amounts of change" are expressed as functions rather than numbers, as they are in the finite difference equation. This is because we want to be able to deal with functions whose domains and ranges are nonnumeric (e.g., lists, sets, relations).[2] Based on this analogy we introduce

Definition 1: Suppose f: $S \rightarrow T$, g: $T \rightarrow T$ and h: $S \rightarrow S$. We define g to be a *(fixed) func-*

1. Note that for *fixed* differences there is no distinction between forward differences and backward differences. This distinction will become important in the study of variable differences.

2. A different notion of functional differences is described in [Paige & Koenig '82]. Their's is addressed to the problem of updating data structures such as sets in an imperative context.

tional difference of f with respect to h if and only if it satisfies $f \circ h = g \circ f$. Furthermore, if the functional difference is unique, we write $g = h \Delta f$, and call g the "h difference of f." We also call g "the change in f with respect to h" or "the change in f given the change h." Thus, when the difference exists and is unique we have

$$f \circ h = (h \Delta f) \circ f \tag{2}$$

2. Existence and Uniqueness of Fixed Differences

Equation 2 defines the functional difference implicitly; to get an explicit definition we need to solve it for $h \Delta f$. This can be accomplished by composing with the inverse of f, f^{-1}, on both sides of the equation to yield:

$$h \Delta f = f \circ h \circ f^{-1}$$

This seems to yield an explicit formula for the functional difference, but it is necessary to be more careful, since it assumes the existence of the (right) inverse f^{-1}. Recall that a right inverse f_r^{-1}, with $f \circ f_r^{-1} = \mathbf{I}_T$, exists only if f is *surjective*. (We write $\mathbf{I}_T$ for the identity $\mathbf{I}: T \rightarrow T$.)

Theorem 1: Let $f: S \rightarrow T$ be a surjection, let $h: S \rightarrow S$, and let $g: T \rightarrow T$ be any function satisfying the equation $f \circ h = g \circ f$. Then $g = f \circ h \circ f_r^{-1}$, where f_r^{-1} is a right inverse of f.

Proof: We compose on the right with f_r^{-1} on both sides of the equation and simplify:

$$g \circ f = f \circ h$$

$$g \circ f \circ f_r^{-1} = f \circ h \circ f_r^{-1}$$

$$g \circ \mathbf{I}_T = f \circ h \circ f_r^{-1}$$

$$g = f \circ h \circ f_r^{-1}$$

□

The preceding theorem assumes a difference exists. The following theorem establishes conditions sufficient for its existence.

Theorem 2: Let f: $S \to T$ be an injection and let h: $S \to S$. The functional equation $f \circ h = g \circ f$ has solutions $f \circ h \circ f_l^{-1}$, for each left inverse f_l^{-1} of f.

Proof: For convenience we represent composition by juxtaposition when no ambiguity will result. Since f is injective it has a left inverse f_l^{-1}, with $f_l^{-1}f = \mathbf{I}_S$. Hence, letting $g = fhf_l^{-1}$, we have

$$gf = (fhf_l^{-1})f = fh(f_l^{-1}f) = fh\mathbf{I}_S = fh$$

Hence, the difference equation is satisfied. □

We observe in passing that none of the preceding results depend on h being either injective or surjective. Thus, they apply to any function h: $S \to S$.[3]

Theorem 2 establishes sufficient conditions for the existence of solutions to the difference equation, namely that f is an injection. The following theorems establish *necessary and sufficient* conditions for the existence and uniqueness of solutions. However, to state them we need the concept of an *equivalence kernel* [MacLane & Birkhoff '67].

Definition 2: The equivalence kernel E_f of a function f: $S \to T$ is the following relation on $S \times S$:

$$E_f = \{ (x, x') \in S \times S \mid fx = fx' \} \tag{3}$$

Thus $(x, x') \in E_f$ if and only if $fx = fx'$. Now we can state our existence theorem:

Theorem 3: A solution g to the difference equation $fh = gf$ exists if and only if $E_f \subseteq E_{fh}$.

Proof: To prove the "if" part, assume that $E_f \subseteq E_{fh}$. Every function f: $S \to T$ can be written as a composition $f = \phi F$ in which ϕ: $S/E_f \to T$ is an injection and F: $S \to S/E_f$ is a surjection. Also, since $E_f \subseteq E_{fh}$, fh can be written as a composition

3. In [MacLennan '86] we introduced at this point the idea of an "isomorphic image" from the calculus of relations. This was used to prove a number of existence theorems for functional differences. Algebraic methods obviate the need for this approach. However, Hasse diagrams are still useful for visualizing functional differences.

$$fh = \pi PF \tag{4}$$

where π: $S/E_{fh} \to T$ is an injection and P: $S/E_f \to S/E_{fh}$ is a surjection. Now we claim that $g = \phi\pi\phi_l^{-1}$ is a solution to the difference equation. This is checked by substitution:

$$gf = (\phi\pi\phi_l^{-1})(\phi F) = \phi\pi F = fh$$

Therefore we have shown that a solution exists if $E_f \subseteq E_{fh}$.

Now to show the "only if" part assume that there is a g such that $fh = gf$. Observe that $E_f \subseteq E_{gf}$ is always true, since g cannot separate values that have already been mapped together by f. But since $fh = gf$, $E_{fh} = E_{gf}$, and therefore $E_f \subseteq E_{fh}$. □

Theorem 4: Suppose $S \neq \mathbf{1}$ and the equation $fh = gf$ has a solution g. Then this solution is unique if and only if f is surjective.

Proof: We prove the contrapositive of the "only if" part. Therefore suppose that f: $S \to T$ is not surjective. Therefore there is a $y \in T$ such that

$$y \notin R = \operatorname{Im} f = \{\, y \mid y = fx \text{ for some } x \in S \,\} \tag{5}$$

We construct a g' different from g that also solves the equation. Let $g'y = a \neq gy$, which is possible so long as $S \neq \mathbf{1}$. Note that since $y \notin R$, $g'f = gf$, and hence $g'f = fh$.

For the "if" part, suppose that both g and g' are solutions and that f is a surjection. Hence,

$$g'f = fh = gf \tag{6}$$

Since f is surjective it has a right inverse; compose with both sides of Eq. 6 to yield:

$$g' = g'ff_r^{-1} = gff_r^{-1} = g$$

Hence, $g = g'$ and the so the solution is unique. □

Obviously it is a critical issue whether f is surjective or not. Therefore we investigate the relationships between the differences of surjective and nonsurjective functions.

Definition 3: The difference equation $fh = gf$ is called *homogeneous* if f is surjective, and *nonhomogeneous* if f is not surjective.

Corollary 4-1: Homogeneous difference equations have at most one solution; nonhomogeneous difference equations do not have unique solutions (i.e., they have no solutions or more than one solution).

Proof: Follows from Theorem 4 and the definition of homogeneous. □

Next we characterize the many solutions of a nonhomogeneous equation as a family of functions derived from an associated homogeneous equation.

Definition 4: If $S \subseteq T$, we use $j_{S \to T}$ for the trivial injection of S into T: $j_{S \to T} x = x$.

Definition 5: Suppose that $f\colon S \to T$, $g\colon T \to T$, $h\colon S \to S$, and $R = \operatorname{Im} f$. Note that f can be written uniquely in the form $f = j_{R \to T} f'$ where $f'\colon S \to R$. Then we call $f'h = gf'$ the *homogeneous equation associated with* the nonhomogeneous equation $fh = gf$.

Definition 6: If $f\colon S_1 \to T$ and $g\colon S_2 \to T$, then we define the *direct sum*, $(f + g)\colon (S_1 + S_2) \to T$, by

$$(f + g)x = \begin{cases} fx & \text{if } x \in S_1 \\ gx & \text{if } x \in S_2 \end{cases} \tag{7}$$

Theorem 5: Let $f\colon S \to T$, $g\colon T \to T$, $h\colon S \to S$, $R = \operatorname{Im} f$, $Q = T - R$ and $j = j_{R \to T}$. Then g is a solution to the nonhomogeneous equation $fh = gf$ if and only if it can be written in the form

$$g = j(g_0 + c) \tag{8}$$

where $g_0\colon R \to R$ is the solution of the associated homogeneous equation, and $c\colon Q \to R$.

Proof: For the "only if" part assume that $fh = gf$ has a solution. Therefore,

$$\mathrm{Im}\ g \subseteq \mathrm{Im}\ gf = \mathrm{Im}\ fh \subseteq \mathrm{Im}\ f = R$$

Hence, $\mathrm{Im}\ g \subseteq R$. This means that g can be written as the direct sum in Eq. 10. It remains to show that g_0 is a solution to the homogeneous equation $f'h = g_0 f'$. Since $fh = gf$, $f = jf'$ and $g = j(g_0 + c)$, we know:

$$jf'h = j(g_0 + c)jf'$$

Since j is an injection, we can compose its inverse on the left of both sides of this equation to yield:

$$f'h = (g_0 + c)jf'$$

Now notice that $(g_0 + c)j = g_0$; therefore $f'h = g_0 f'$ and we have proved that g_0 is a solution. In fact it is the unique solution because the equation is homogeneous.

To prove the "if" part we assume that there is a g_0 such that $f'h = g_0 f'$. Let $g = j(g_0 + c)$; our goal is to show $fh = gf$. Noting again that $(g_0 + c)j = g_0$, derive:

$$\begin{aligned} fh &= jf'h \\ &= jg_0 f' \\ &= j(g_0 + c)jf' \\ &= gf \end{aligned}$$

Hence g solves the nonhomogeneous equation. □

Corollary 5-1: A nonhomogeneous functional difference can be written uniquely in terms of the associated homogeneous difference:

$$h \,\Delta\, jf = j(h \,\Delta\, f + c) \tag{9}$$

where $f: S \to R$ is surjective, and $h: S \to S$, $j = j_{R \to R+Q}$ and $c: Q \to R$. This is call the *general*

form of the solution of the nonhomogeneous equation.

Proof: Follows from the preceding proof. □

3. Examples of Differences

In this section we give several examples of functional differences. We begin with numerical functions, since they are most familiar. Let $\mathbb{N}$ be the natural numbers and let $\sigma: \mathbb{N} \to \mathbb{N}$ be the successor function. We use the presection and postsection notations [Wile '73]: $[a +]\ x = a + x$, $[- b]\ x = x - b$, etc. Also, we let '$\uparrow$' represent exponentiation, $a \uparrow n \ = \ a^n$.

Definition 7: We write '$(\text{power}_a\ f)\ n$' for the n^{th} power of function f applied to initial value a: $(\text{power}_a\ f)\ n \ = \ f^n\ a$. This is defined recursively:

$$\begin{aligned} (\text{power}_a\ f)\ 0 \ &= \ a \\ (\text{power}_a\ f)\ (n+1) \ &= \ f\ [(\text{power}_a\ f)\ n], \ \text{ for } n \geqslant 0 \end{aligned} \tag{10}$$

We call $\text{power}_a\ f$ 'the power from a of f'. If $f: S \to S$ and $a \in S$, then it is clear that $\text{power}_a\ f: \mathbb{N} \to S$.

Theorem 6: The difference of the power (from a) of f with respect to successor is f,

$$\sigma\ \Delta\ \text{power}_a\ f \ = \ f, \tag{11}$$

provided that $\text{power}_a\ f$ is surjective.

Proof: This follows directly from the definition of 'power':

$$(\text{power}_a\ f)\ (\sigma\ n) \ = \ f\ [(\text{power}_a\ f)\ n]$$

Therefore, $(\text{power}_a\ f) \circ \sigma \ = \ f \circ (\text{power}_a\ f)$. □

Theorem 7:

$$\sigma\ \Delta\ \text{power}_a\ f \ = \ fj + jc$$

$$(12)$$

where $j = j_{R \to S}$, $R = \mathrm{Im}\ (\mathrm{power}_a\ f)$ and c: $(S - R) \to R$.

Proof: Let π: $R \to R$ be $\mathrm{power}_a\ f$ with its range restricted to R. Thus,

$$j\pi \;=\; \mathrm{power}_a\ f \tag{13}$$

Since π is surjective, $\sigma \,\Delta\, \pi$ exists. Observe that

$$\mathrm{Im}\ fj \;=\; \mathrm{Im}\ [f(\mathrm{power}_a f)] \;=\; \mathrm{Im}\ [(\mathrm{power}_a f)\sigma] \;\subseteq\; \mathrm{Im}\ (\mathrm{power}_a f) \;=\; R$$

Therefore, if f's domain is restricted to R then its image is a subset of R. Hence there is a ϕ: $R \to R$ such that

$$j\phi \;=\; fj \tag{14}$$

The definition of $\mathrm{power}_a\ f$ is equivalent to:

$$j\pi\sigma \;=\; (\mathrm{power}_a f)\sigma \;=\; f(\mathrm{power}_a f) \;=\; fj\pi \;=\; j\phi\pi$$

Hence, $j\pi\sigma = j\phi\pi$. Composing a left inverse of j with both sides of the equation yields

$$\pi\sigma \;=\; \phi\pi \tag{15}$$

Since π is surjective $\sigma \,\Delta\, \pi = \phi$. Then, by Cor. 5-1,

$$\begin{aligned}
\sigma \,\Delta\, \mathrm{power}_a\ f &= \sigma \,\Delta\, j\pi \\
&= j(\sigma \,\Delta\, \pi + c) \\
&= j(\phi + c) \\
&= j\phi + jc \\
&= fj + jc
\end{aligned}$$

□

Now observe that

$$\lfloor a\ +\rfloor = \mathrm{power}_a\ \sigma$$

$$\lfloor a\ -\rfloor = \mathrm{power}_a\ \sigma^{-1}$$

$$\lfloor -\ a\rfloor = \mathrm{power}_{-a}\ \sigma$$

$$\lfloor a\ \times\rfloor = \mathrm{power}_0\ \lfloor a\ +\rfloor$$

$$\lfloor a\ \uparrow\rfloor = \mathrm{power}_1\ \lfloor a\ \times\rfloor$$

Notice that the last two of these are not surjective. The differences of these functions follow from the theorem:

$$\sigma\ \Delta\ \lfloor a\ +\rfloor = \sigma$$

$$\sigma\ \Delta\ \lfloor a\ -\rfloor = \sigma^{-1}$$

$$\sigma\ \Delta\ \lfloor -\ a\rfloor = \sigma$$

$$\sigma\ \Delta\ \lfloor a\ \times\rfloor = \lfloor a\ +\rfloor j + jc \quad \text{where} \quad j = j_{\mathrm{Im}\,\lfloor a\ \times\rfloor \to \mathbb{N}}$$

$$\sigma\ \Delta\ \lfloor a\ \uparrow\rfloor = \lfloor a\ \times\rfloor j + jc \quad \text{where} \quad j = j_{\mathrm{Im}\,\lfloor a\ \uparrow\rfloor \to \mathbb{N}}$$

4. Properties of Differences

We develop a number of simple, useful properties of functional differences.

Theorem 8: The functional difference of the identity $\mathbf{I}$: $S \to S$ with respect to any function h: $S \to S$ is that function: $h\ \Delta\ \mathbf{I} = h$.

Proof: Observe that $g = h$ satisfies $\mathbf{I}h = g\mathbf{I}$ and that $\mathbf{I}$ is surjective. □

Theorem 9: A functional difference of h: $S \to S$ with respect to itself is itself. Further, if h is surjective then $h\ \Delta\ h = h$.

Proof: Observe $g = h$ satisfies $hh = gh$. □

Corollary 9-1: A difference of a power of a function with respect to that function is that

function. Also, if h^n is surjective then $h \,\Delta\, h^n = h$.

Proof: Let $g = h$ in $h^n h = gh^n$. □

Theorem 10: A (left) inverse of a difference of a function with respect to a surjection is a difference of the function with respect to a (right) inverse of the surjection. That is, if $fh = gf$ then

$$fh_r^{-1} = g_l^{-1}f \tag{16}$$

If f is surjective, then $(h \,\Delta\, f)_l^{-1} = h_r^{-1} \,\Delta\, f$.

Proof: Suppose h is a surjection and there is a g satisfying $fh = gf$. Suppose that g_l^{-1} is a left inverse of g. Then compose g_l^{-1} with both sides of $fh = gf$. This yields $g_l^{-1}fh = f$. Since h is surjective it has a right inverse h_r^{-1}. Composing with h_r^{-1} on both sides of $f = g_l^{-1}fh$ yields: $fh_r^{-1} = g_l^{-1}f$. Hence g_l^{-1} is a functional difference of f with respect to h_r^{-1}. □

It will usually be clear to drop the l and r subscripts and write $(h \,\Delta\, f)^{-1} = h^{-1} \,\Delta\, f$.

Theorem 11: A difference of a composition of two functions with respect to a third is a difference of the first function with respect to a difference of the second with respect to the third. That is, if ϕ is a difference of g with respect to h and ψ is a difference of f with respect to ϕ, then ψ is also a difference of fg with respect to h. Also, if f and g are surjective, $h \,\Delta\, fg = (h \,\Delta\, g) \,\Delta\, f$.

Proof: The derivation is direct. Suppose $gh = \phi g$ and $f\phi = \psi f$. Then

$$(fg)h = f(gh) = f(\phi g) = (f\phi)g = (\psi f)g = \psi(fg).$$

Hence, $(fg)h = \psi(fg)$. If f and g are surjective then the differences are unique. □

Corollary 11-1: If g is surjective and f is any function such that the differences exist, then $h \,\Delta\, fg = (h \,\Delta\, g) \,\Delta\, f$.

Proof: Suppose that $f = j\phi$, with ϕ surjective. We expand the general form for the difference:

$$\begin{aligned}
h \Delta fg &= h \Delta j\phi g \\
&= j(h \Delta \phi g + c) \\
&= j[(h \Delta g) \Delta \phi + c] \\
&= (h \Delta g) \Delta j\phi \\
&= (h \Delta g) \Delta f
\end{aligned}$$

The third line above follows from the preceding theorem. □

Theorem 12: The difference of a bijection with respect to a composition of functions is the composition of the differences of the bijection with respect to each of the functions:

$$gh \Delta f = (g \Delta f)(h \Delta f) \tag{17}$$

Proof: Since f is bijective it has the two-sided inverse f^{-1}; hence by Theorem 1:

$$\begin{aligned}
gh \Delta f &= fghf^{-1} \\
&= fg(f^{-1}f)hf^{-1} \\
&= (fgf^{-1})(fhf^{-1}) \\
&= (g \Delta f)(h \Delta f)
\end{aligned}$$

□

The preceding theorem provides a kind of chain rule for evaluating differences of bijections. In the following theorem we extend it to any functions with differences.

Theorem 13: A functional difference of a function with respect to a composition of functions is the composition of the difference of the first function with respect to each of the other functions. That is, if ϕ is a difference of f with respect to g and ψ is a difference of f with respect to h, then $\phi\psi$ is a difference of f with respect to gh.

Proof: Since $fg = \phi f$ and $fh = \psi f$ we may derive:

$$f(gh) = (fg)h = (\phi f)h = \phi(fh) = \phi(\psi f) = (\phi\psi)f$$

Hence $f(gh) = (\phi\psi)f$, and $\phi\psi$ is a functional difference of f with respect to gh. □

Corollary 13-1: If f is any function, $gh \ \Delta\ f = (g \ \Delta\ f)(h \ \Delta\ f)$, provided the differences exist. Note that if f is not surjective, this corollary asserts the identity of the *general forms* of the differences.

Proof: If the differences exist, then they may be written

$$g \ \Delta\ f = j(g\Delta\phi + c), \quad h \ \Delta\ f = j(h\Delta\phi + c')$$

where $j\phi = f$. Now observe the product:

$$\begin{aligned}(g \ \Delta\ f)(h \ \Delta\ f) &= j(g\Delta\phi + c)j(h\Delta\phi + c') \\ &= j(g\Delta\phi)(h\Delta\phi + c') \\ &= j[(g\Delta\phi)(h\Delta\phi) + (g\Delta\phi)c'] \\ &= j[(g\Delta\phi)(h\Delta\phi) + c''] \\ &= j(gh\Delta\phi + c'')\end{aligned}$$

The last line follows by the preceding theorem; it is the general form of $gh \ \Delta\ f$. □

Corollary 13-2: A difference of a function with respect to the nth power of a function is the nth power of a difference: $h^n \ \Delta\ f = (h \ \Delta\ f)^n$.

Proof: This is an inductive application of the previous theorems. □

We use a product notation for compositions:

$$\prod_{i=1}^{n} F_i = F_1 \circ F_2 \circ \cdots \circ F_{n-1} \circ F_n$$

Using this notation we can express

Corollary 13-3: A difference of a function with respect to a product of functions is the product of differences with respect to each of those functions:

$$(\prod_{i=1}^{n} h_i) \Delta f = \prod_{i=1}^{n} (h_i \Delta f) \tag{18}$$

Proof: An inductive application of the theorems. □

Corollary 13-4: If $x_n = h^n x_0$, then $f x_n = (h \Delta f)^n (f x_0)$, where $h \Delta f$ is any difference of f with respect to h. That is, if $y_i = f x_i$ then $y_n = (h \Delta f)^n y_0$. Equivalently, $y = \text{power}_{y_0} h \Delta f$.

Proof: This is an induction based on $y_n = f(h x_{n-1}) = (h \Delta f)(f x_{n-1}) = (h \Delta f) y_{n-1}$. □

Corollary 13-5: If $x_n = \left(\prod_{i=1}^{n} H_i\right) x_0$ and $y_i = F x_i$ then $y_n = \left(\prod_{i=1}^{n} H_i \Delta F\right) y_0$.

Proof: This is just an application of Cor. 13-3. □

This theorem tells us how to use functional differences to get from $f x_0$ to $f x_n$, provided x_n is reachable from x_0. It is a functional difference analogue of Taylor's Theorem.

We define[4] 'Δf' so that $(\Delta f) h = h \Delta f$. Since Δf leaves h unspecified, we call it the *indefinite functional difference* of f. One difficulty with this notation is that $(\Delta f) h$ is defined only if $E_f \subseteq E_{fh}$ (Theorem 3), and is single valued only if f is surjective (Theorem 4). To avoid these problems we restrict the notation Δf to the case in which f is bijective, since then we are guaranteed that the indefinite difference exists (Theorems 2 and 4).

Definition 8: The *indefinite functional difference* Δf: $(S \to S) \to (T \to T)$ is defined:

$$(\Delta f) h = h \Delta f \tag{19}$$

for any bijection f: $S \leftrightarrow T$. Note that the signature of Δ is:

4. This is simply the postsection notation from [Wile '73].

$$\Delta\text{: Bijec }(S,\ T) \to [(S \to S) \to (T \to T)] \tag{20}$$

where Bijec $(S,\ T)$ is the category of all bijections from S to T,

Theorem 14: The indefinite difference of the composition of bijections is the composition of the indefinite differences of the bijections:

$$\Delta\,(f \circ g) \;=\; (\Delta f) \circ (\Delta g)$$

Proof: By Theorem 11:

$$\begin{aligned}(\Delta\ fg)h &= h\ \Delta\ fg \\ &= (h\ \Delta\ g)\ \Delta\ f \\ &= (\Delta\ f)(h\ \Delta\ g) \\ &= \Delta f\ [(\Delta\ g)h]\end{aligned}$$

Hence, $\Delta\ fg = \Delta f \circ \Delta g$. □

The preceding theorem is a kind of chain rule for functional differences. It leads to

Corollary 14-1: The indefinite difference of the nth power of a bijection is the nth power of the indefinite difference of that bijection: $\Delta\ f^n \;=\; (\Delta f)^n$.

Proof: This is just the inductive extension of the previous theorem. □

Corollary 14-2: The indefinite difference of the product of bijections is the product of the indefinite differences of those bijections:

$$\Delta \left(\prod_{i=1}^{n} f_i \right) \;=\; \prod_{i=1}^{n} \Delta f_i$$

That is, $h\ \Delta \left(\prod_{i=1}^{n} f_i \right) \;=\; \left(\prod_{i=1}^{n} \Delta f_i \right)(h)$

Proof: This also follows inductively from the theorem. □

5. Recursive Programs

Let T^* be the type of all LISP-like lists whose members are of type T. Consider the following equations, which define the function length: $T^* \to \mathbb{N}$ on lists:

$$
\begin{aligned}
\text{length nil} &= 0 \\
\text{length } (x : y) &= 1 + \text{length } y
\end{aligned}
\tag{21}
$$

(Here '$x : y$' denotes the result of prefixing x on the list y — the LISP 'cons' operation.) The second equation is a homogeneous functional difference equation, as can be seen by writing it in the form:

$$
\text{length} \circ [x :] = \sigma \circ \text{length}
\tag{22}
$$

(Note that $[x :]: T^* \to T^*$.) Hence it is easy to see that the change in length with respect to prefixing is the successor:

$$
[x :] \,\Delta\, \text{length} = \sigma
\tag{23}
$$

On the other hand, if we were to define length recursively, we would write something like this:

$$
\text{length } L = \begin{cases} 0, & \text{if } L = \text{nil} \\ 1 + \text{length (rest } L), & \text{if } L \neq \text{nil} \end{cases}
\tag{24}
$$

This corresponds to the equations

$$
\begin{aligned}
\text{length nil} &= 0 \\
\text{length } L &= 1 + \text{length (rest } L), \text{ for } L \neq \text{nil}
\end{aligned}
\tag{25}
$$

The second equation here is also a sort of difference equation, but it does not fit our earlier form. Written in terms of compositions it is:

$$
\text{length} \circ j_+ = \sigma \circ \text{length} \circ \text{rest}
\tag{26}
$$

where we have composed length with j_+ to restrict its domain to nonnull lists. The function

$j_+ = j_{T^+ \to T^*}$ injects the type of nonnull sequences T^+ into the type of (possibly null) sequences T^*. Note also that we assume 'rest' is a total function, rest: $T^+ \to T^*$. Hence the domains and ranges of the functions match as required by the compositions.

What is the relationship between the two difference equations satisfied by length? Consider the first difference equation (22):

$$\text{length} \circ [x :] = \sigma \circ \text{length} \tag{22}$$

We would like to compose with the inverse of $[x :]$ on both sides. Unfortunately we can't do this because $[x :]$ is not a surjection, so it's not right invertible. Therefore we will have to use an alternate approach.

Let $xT^* \subset T^+$ represent the type of all sequences that begin with x. Since the meaning of $[x :]$ is to put x on the front of its argument, we will consider the bijection π_x: $T^* \to xT^*$; this operation is $[x :]$ with its range restricted to xT^*. Hence $[x :] = j_+ j_x \pi_x$, where $j_x = j_{xT^* \to T^+}$. Since π_x is a bijection, it inverse π_x^{-1}: $xT^* \to T^*$ exists (and is a bijection). The meaning of π_x^{-1} is to take x off the front of a sequence that begins with x. On the other hand 'rest' takes the first thing off the front of its argument no matter what it is. Hence π_x^{-1} is like 'rest' except that it's defined only on lists whose first element is x. That is, π_x^{-1} is a restriction of 'rest' to the domain xT^*, $\pi_x^{-1} = \text{rest} \circ j_x$.

Now we make two simple observations. First, the type of all nonnull lists is the direct sum, for all x, of the nonnull lists that begin with x:

$$T^+ = \sum_{x \in T} xT^*$$

Second, the 'rest' function, which deletes the first element of a list no matter what it is, is the direct sum of all the functions π_x^{-1}, which delete x from the front of a list:

$$\text{rest} = \text{rest} \circ \sum_{x \in T} j_x = \sum_{x \in T} \text{rest} \circ j_x = \sum_{x \in T} \pi_x^{-1}$$

It is now easy to show the two difference equations are equivalent.

Theorem 15: Suppose that

$$\text{length} \circ [x:] = \sigma \circ \text{length} \tag{27}$$

is true for all x. Then

$$\text{length} \circ j_+ = \sigma \circ \text{length} \circ \text{rest} \tag{28}$$

The converse also holds.

Proof: To prove the first (Eq. 27) implies the second (Eq. 28) we have:

$$\begin{aligned}\sigma \circ \text{length} &= \text{length} \circ [x:] \\ &= \text{length} \circ j_+ \circ j_x \circ \pi_x\end{aligned}$$

Compose on both sides with π_x^{-1} to yield:

$$\sigma \circ \text{length} \circ \pi_x^{-1} = \text{length} \circ j_+ \circ j_x$$

Now derive:

$$\begin{aligned}\text{length} \circ j_+ &= \text{length} \circ j_+ \circ \sum_{x \in T} j_x \\ &= \sum_{x \in T} (\text{length} \circ j_+ \circ j_x) \\ &= \sum_{x \in T} (\sigma \circ \text{length} \circ \pi_x^{-1}) \\ &= \sigma \circ \text{length} \circ \sum_{x \in T} \pi_x^{-1} \\ &= \sigma \circ \text{length} \circ \text{rest}\end{aligned}$$

This is Eq. 28.

To prove the second (Eq. 28) implies the first (Eq. 27) we restrict both sides to xT^*:

$$\text{length} \circ j_+ \circ j_x = \sigma \circ \text{length} \circ \text{rest} \circ j_x$$

Recalling that rest $\circ\ j_x = \pi_x^{-1}$:

$$\text{length} \circ j_+ \circ j_x = \sigma \circ \text{length} \circ \pi_x^{-1}$$

Now compose with π_x on both sides to obtain Eq. 27. □

6. Definition and Existence of Integral

In this section we consider a *functional integration* operation that is inverse to the functional difference. That is, given functions $g\colon T \to T$ and $h\colon S \to S$ we want to find an $f\colon S \to T$ that satisfies the functional difference equation $fh = gf$.

In general there may be many functions satisfying this relationship. Since this implies that the solution to a functional difference equation is often *underdetermined* by the equation, to determine a particular solution it's necessary to specify a *boundary function* b contained in the solution. Thus the solution to the equation is required to be an extension of the boundary function (i.e., $b = fj_{D \to S}$, where $D \subseteq S$).

Definition 9: Suppose $D \subseteq S$. Let arbitrary functions $g\colon T \to T$, $h\colon S \to S$ and $b\colon D \to T$ be given. If there is a function $f\colon S \to T$, with $b = fj_{D \to S}$, satisfying the equation

$$f \circ h = g \circ f$$

then we call f the *definite functional integral* with respect to h, from b, of g. We write f:

$$h\ \Phi_b\ g$$

This is read "the h integral from b of g."

Theorem 16: If the definite functional integral exists, then it satisfies the equation:

$$\left(h\ \Phi_b\ g\right) \circ h = g \circ \left(h\ \Phi_b\ g\right)$$

Proof: Follows immediately from definition. □

Next we explore some of the conditions under which functional integrals exist.

Definition 10: S is *generated from D by h*: $S \to S$ if and only if

$$S = \bigcup_{n \geqslant 0} h^n[D] \tag{29}$$

(The notation $h^n[D]$ denotes the image of D under h^n.) This is equivalent to S satisfying the recursive equation

$$S = D \cup h[S] \tag{30}$$

Note that if S is generated by h, then h is surjective.

If the generation of S loses information that is required by g, then the functional integral cannot be computed; this is formalized in the following theorems. To state them we introduce the class of functions for which functional integrals will be shown to exist:

Definition 11: Suppose g: $T \to T$, b: $D \to T$, and S is generated from D by h: $S \to S$. We say that the functions $g^k b$ *ignore the generation of* S if and only if, for all x_0, $x_0{}' \in D$, m, $n \geqslant 0$,

$$h^m x_0 = h^n x_0{}' \Rightarrow g^m b x_0 = g^n b x_0{}' \tag{31}$$

That is, the functions $g^k b$ don't care if an element of S can be generated in two or more ways.

Lemma 17-1: Suppose b: $D \to T$, f: $S \to T$, g: $T \to T$, $b = f j_{D \to S}$, and S is generated from D by h. If $fh = gf$ has a solution then the functions $g^k b$ ignore the generation of S.

Proof: Let $x \in S$ and $j = j_{D \to S}$. Suppose $h^m j x_0 = x = h^n j x_0{}'$ for x_0, $x_0{}' \in D$. (Note that for $x_0 \in D$, $j x_0 = x_0$.) By Cor. 13-4,

$$fx = fh^m j x_0 = g^m f j x_0 = g^m b x_0$$

$$fx = fh^n j x_0{}' = g^n f j x_0{}' = g^n b x_0{}'$$

Hence, $g^m b x_0 = g^n b x_0{}'$. □

Next we give a formula for the h integral from b of g. It is defined in terms of the graph of the function.

Definition 12: If b: $D \to T$, h: $S \to S$, g: $T \to T$, and $D \subseteq S$, define the relation Φ_{hbg} as follows:

$$\Phi_{hbg} = \{ (h^n x_0, g^n b x_0) \mid x_0 \in D, n \geqslant 0 \} \tag{32}$$

Lemma 17-2: If the $g^k b$ ignore the generation of S, then there is a unique function ϕ_{hbg} such that

$$\text{graph } \phi_{hbg} = \Phi_{hbg} \tag{33}$$

where graph f is the graph of the function f (i.e., the set of (x, y) such that $fx = y$.). Further, this function satisfies

$$\phi_{hbg}(h^n x_0) = g^n b x_0 \quad \text{for } x_0 \in D \tag{34}$$

Proof: Clearly there is at most one function ϕ: $S \to T$ having a given graph. Thus it is necessary to show that $\phi = \phi_{hbg}$ is defined and single valued for every member of S. Since S is generated by h from D, each $x \in S$ can be written $x = h^n x_0$ for some $x_0 \in D$, $n \geqslant 0$. Hence, from the definition of Φ_{hbg} we see ϕx is defined and

$$\phi x = g^n b x_0, \text{ for } x = h^n x_0 \tag{35}$$

To show ϕx is single valued, assume $h^m x_0 = h^n x_0{}'$. By Eq. 35,

$$\phi x = g^m b x_0, \quad \phi x = g^n b x_0{}'$$

But the $g^k b$ ignore the generation of S, so $g^m b x_0 = g^n b x_0{}'$. Hence, ϕ is single valued. □

Next we prove that ϕ_{hbg} is a solution provided the $g^k b$ ignore generation.

Lemma 17-3: If the $g^k b$ ignore the generation of S, then

$$\phi_{hbg} = g\phi_{hbg} \tag{36}$$

Proof: For an arbitrary $x \in S$ we show $\phi hx = g\phi x$, where $\phi = \phi_{hbg}$. Since S is generated from D by h, write $x = h^n jx_0$, where $j = j_{D\to S}$. Then,

$$\begin{aligned} \phi hx &= \phi h^{n+1} jx_0, \text{ since } x = h^n jx_0 \\ &= g^{n+1} bx_0, \text{ by Lemma 16-2} \\ &= gg^n bx_0 \\ &= g\phi x, \text{ by Eq. 35} \end{aligned}$$

□

We can now prove our principal existence theorem:

Theorem 17: Let $h: S \to S$, $b: D \to T$, $g: T \to T$, and suppose S is generated from D by h. Then $fh = gf$ has a solution if and only if the $g^k b$ ignore the generation of S.

Proof: This follows from Lemmas 17-1 and 17-3. □

Our next goal is to show that ϕ_{hbg} is the unique solution of the difference equation.

Lemma 18-1: Suppose $h: S \to S$, $b: D \to T$, $g: T \to T$, and $f: S \to T$. If the $g^k b$ ignore the generation of S and f is a solution to $fh = gf$, then $f = \phi_{hbg}$.

Proof: Suppose $x = h^n x_0$ for $x_0 \in D$. By Cor. 13-4,

$$fx = fh^n jx_0 = g^n fjx_0 = g^n bx_0 = \phi_{hbg} x$$

□

Theorem 18: Suppose $h: S \to S$, $b: D \to T$, $g: T \to T$. If S is generated from D by h, but the $g^k b$ ignore this generation, then

$$h \, \Phi_b \, g \;=\; \phi_{hbg} \tag{37}$$

Proof: This follows from Lemmas 17-3 and 18-1. □

7. Properties of Integrals

The following corollaries follow easily from the preceding theorems and from the properties of differences in section 4.

Corollary 18-1:

$$\left(h \, \Phi_b \, g\right) (h^n x_0) \;=\; g^n b x_0, \text{ for } x_0 \in D \tag{38}$$

Proof: Follows from Theorem 18 and Lemma 17-2. □

Corollary 18-2: $h \, \Phi_{\mathbf{I}_D} \, h \;=\; \mathbf{I}$.

Proof: Follows from Thm. 8. □

Corollary 18-3: $h \, \Phi_{hj_{D,S}} \, h \;=\; h$.

Proof: Follows from Thm. 9. □

Corollary 18-4: $h \, \Phi_{h^n j_{D,S}} \, h \;=\; h^n$.

Proof: Follows from Cor. 9-1. □

Corollary 18-5: Suppose $h\colon R \to R$, $\phi\colon S \to S$, $\psi\colon T \to T$, $b\colon S_0 \to T$, $c\colon R_0 \to S$, where $S_0 \subseteq S$ and $R_0 \subseteq R$. Then, if the $\psi^k b$ ignore the generation of S by ϕ, and the $\phi^k c$ ignore the generation of R by h, and the $\psi^k bc$ ignore the generation of R by h, then

$$(\phi \, \Phi_b \, \psi)(h \, \Phi_c \, \phi) = h \, \Phi_{bjc} \, \psi \tag{39}$$

where $j = j_{S_0 \to S}$

Proof: Follows from Thm. 13. □

The following two theorems are the **Fundamental Theorems** relating fixed differences and integrals.

Theorem 19: Suppose $f \colon S \to T$, $h \colon S \to S$ and $b = fj_{D \to S}$, where S is generated from D by h. If $E_f \subseteq E_{fh}$ and f is surjective,

$$h \, \Phi_b \, (h \, \Delta \, f) = f \tag{40}$$

Proof: Under the stated conditions the difference equation $fh = gf$ has a unique solution $g = h \, \Delta \, f$. To show that f is the functional integral of g with respect to h we must show that the $g^k b$ ignore the generation of S by h. Therefore suppose $h^m j x_0 = h^n j x_0{}'$, where $j = j_{D \to S}$. Then,

$$\begin{aligned} g^m b x_0 &= g^m f j x_0, \text{ by definition of } b \\ &= f h^m j x_0, \text{ Cor. 13-4} \\ &= f h^n j x_0{}', \text{ by hypothesis} \\ &= g^n f j x_0{}', \text{ Cor. 13-4} \\ &= g^n b x_0, \text{ by definition of } b \end{aligned}$$

Hence f is the h integral from b of g: $h \, \Phi_b \, g = f$. Combining with $g = h \, \Delta \, f$ yields the desired result. □

Theorem 20: Suppose $h \colon S \to S$, $b \colon D \to T$, $g \colon T \to T$, and S is generated from D by h. Then,

$$h \,\Delta \left(h \,\Phi_b\, g\right) = g \tag{41}$$

if and only if T is generated from $b[D]$ by g.

Proof: Under the stated conditions the integral $f = h \,\Phi_b\, g$ exists. The difference will exist and be unique if f is surjective and $E_f \subseteq E_{fh}$. To determine the conditions under which f is surjective we compute a formula for the image of f:

$$\begin{aligned}
\mathrm{Im}\, f &= \mathrm{rng}\, \Phi_{hbg} \\
&= \mathrm{rng}\, \{ \, (h^n x_0,\, g^n b x_0) \mid x_0 \in D,\, n \geqslant 0 \, \} \\
&= \{ \, g^n b x_0 \mid x_0 \in D,\, n \geqslant 0 \, \} \\
&= \bigcup_{n \geqslant 0} \mathrm{Im}\, g^n b
\end{aligned}$$

Now observe that $T = \bigcup_{n \geqslant 0} \mathrm{Im}\, g^n b$ if and only if T is generated from $b[D]$ by g.

Next we must show $E_f \subseteq E_{fh}$. Therefore assume $fx = fx'$; that is, $fh^m x_0 = fh^n x_0'$. By Lem. 17-2, $fh^m x_0 = g^m b x_0$ and $fh^n x_0' = g^n b x_0'$. Hence, $g^m b x_0 = g^n b x_0'$. Now derive:

$$\begin{aligned}
fhx &= fh^{m+1} x_0 = g^{m+1} b x_0 \\
&= g(g^m b x_0) = g(g^n b x_0') \\
&= g^{n+1} b x_0' = fh^{n+1} x_0' \\
&= fhx'
\end{aligned}$$

Hence $E_f \subseteq E_{fh}$ and the difference is unique. □

The following corollary shows that the difference of a nonsurjective integral is the integrand plus a remainder term.

Corollary 20-1: If T is not generated from $b[D]$ by g, then

$$h \,\Delta \left(h \,\Phi_b\, g\right) \;=\; gj + jc \tag{42}$$

where $h\colon S \to S$, $b\colon D \to T$, $g\colon T \to T$, $j = j_{R \to T}$, $c\colon (T - R) \to R$, and $R = \mathrm{Im}\,(j \,\Phi_b\, g)$.

Proof: Let $Q = T - R$. Since $R = \mathrm{Im}\,(h \,\Phi_b\, g)$ we can write $g = jq + r$ where $q\colon R \to R$ and $r\colon Q \to T$. Note that q is surjective and R is generated from $b[D]$ by q. Hence $h \,\Phi_a\, q$ exists, where $a\colon D \to R$, $ja = b$. Note that $h \,\Phi_b\, g = j(h \,\Phi_a\, q)$ and apply Cor. 5-1:

$$\begin{aligned} h \,\Delta\, (h \,\Phi_b\, g) &= h \,\Delta\, j(h \,\Phi_a\, q) \\ &= j[h \,\Delta\, (h \,\Phi_a\, q) + c] \\ &= j(q + c) \end{aligned}$$

since $h \,\Phi_a\, q$ is surjective. Now observe that

$$gj = (jq + r)j \;=\; jq$$

Hence the difference of the integral is $gj + jc$. □

The functional integral permits a different characterization of the power functional:

Theorem 21: The σ integral from $0 \to a$ of f is the power from a of f:

$$\sigma \,\Phi_{0 \to a}\, f \;=\; \mathrm{power}_a\, f \tag{43}$$

Proof: Note that $\mathbb{N}$ is generated from $\{0\}$ by σ; to show the existence of the integral we must show that the $f^k \circ (0 \to a)$ ignore this generation. Hence we need

$$\sigma^m 0 = \sigma^n 0 \;\Rightarrow\; f^m(0 \to a)0 \;=\; f^n(0 \to a)0$$

The left-hand side of this implication is equivalent to $m = n$, which obviously implies the right-hand side. Therefore, the functional integral exists and satisfies the following equation:

$$\left(\sigma \,\Phi_{0 \to a}\, f\right)\sigma \;=\; f\left(\sigma \,\Phi_{0 \to a}\, f\right)$$

(44)

But this is just the difference equation which defines the power function. □

The following corollary is Thm. 7, but obtained via the integral.

Corollary 21-1:

$$\sigma \, \Delta \, \mathrm{power}_a \, f = fj + jc \tag{45}$$

where $j = j_{R \to S}$ and $R = \mathrm{Im}\,(\mathrm{power}_a \, f)$.

Proof: By the theorem:

$$\sigma \, \Delta \, \mathrm{power}_a \, f = \sigma \, \Delta \, (\sigma \, \Phi_{0 \to a} \, f) = fj + jc$$

□

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